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2D and 3D upper bound solutions for tunnel excavation using 'elastic' flow fields

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SUMMARY

This paper presents 2D and 3D upper bound solutions for the problem of tunnel excavation in soft ground. The solution invokes the use of incompressible flow fields derived from the theory of elasticity and the concept of sinks and sources. Comparison is made with previously published results. For some geometries the current calculation results in lower (better) upper bound values; however, the results were generally close to previously published values. Copyright © 2007 John Wiley & Sons, Ltd.

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INTRODUCTION

Tunnel stability refers here to the stability of an unlined heading area prior to lining placement. Davis *et al.* [1] dealt thoroughly with the problem by suggesting design values derived by use of the upper and lower bound theorems of plasticity. Unless a finite element tool for limit analysis is used (e.g. Sloan [2]), some assumption of the mode of failure is required if an upper bound analysis is to be conducted. For 2D problems, establishing an admissible collapse mode is an uncomplicated task requiring some intuition and 'back-of-the-envelope' calculations (see Davis *et al.* [1] for simple mechanisms). On the other hand, for 3D problems, the task of formulating an admissible velocity field from intuition becomes far more difficult, if not impossible.

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Any method that helps in establishing an admissible field for upper bound calculations is valuable, especially in the case of 3D problems (e.g. Puzrin and Randolph [3]).

Osman *et al.* [4] suggested an interesting approach for calculating upper bound values. They adopted the commonly used Gaussian settlement trough as an indicator of the shape of the velocity field near the ground surface in the event of collapse. By using the continuity equation for incompressible material, in conjunction with the settlement trough as the top boundary condition, they obtained an admissible velocity field for upper bound calculations for the plane strain collapse of a 2D cavity. The values arising from their calculation method were found to be in good agreement with previously published solutions. The current work uses a somewhat similar concept; however, instead of using the Gaussian curve as an indicator and integrating the differential equation of continuity, an admissible velocity field is obtained directly from elasticity theory. This process is also straightforwardly applied to the 3D case of an unsupported tunnel heading.

2D ANALYSIS—PLANE STRAIN CASE

The upper bound theorem states that if a work calculation is performed for a kinematically admissible collapse mechanism, then the deduced load will be either higher than, or equal to, that which will cause collapse. For Tresca's yield criterion, an admissible collapse field will be any velocity field which will entail conditions of incompressibility, and satisfies any imposed velocity boundary condition. Shear bands and slip lines are allowed.

A similar logic to that of Osman *et al.* [4] is employed in the current work except that the velocity fields are based on analytical solutions, rather than on the empirical Gaussian curve. A plastic flow field that is proportional to a deformation field based on elasticity (e.g. Verruijt and Booker [5]) is first generated. For the case of incompressible soil ($\nu = 0.5$), Verruijt and Booker's solution may be written as

$$u_x = -\varepsilon r_0^2\left(\frac{x}{x^2+(z-h)^2}+\frac{x}{x^2+(z+h)^2}-4x\,\frac{z(z+h)}{(x^2+(z+h)^2)^2}\right)$$
$$+\delta r_0^2\left(\frac{x(x^2-(z-h)^2)}{(x^2+(z-h)^2)^2}+\frac{x(x^2-(z+h)^2)}{(x^2+(z+h)^2)^2}-4xh\,\frac{z(x^2-3(z+h)^2)}{(x^2+(z+h)^2)^3}\right)$$

$$u_z = -\varepsilon r_0^2\left(\frac{z-h}{x^2+(z-h)^2}+\frac{z+h}{x^2+(z+h)^2}-\frac{2(z+h)}{x^2+(z+h)^2}+\frac{2z(x^2-(z+h)^2)}{(x^2+(z+h)^2)^2}\right)$$
$$+\delta r_0^2\left(\frac{(z-h)(x^2-(z-h)^2)}{(x^2+(z-h)^2)^2}+\frac{(z+h)(x^2-(z+h)^2)}{(x^2+(z+h)^2)^2}-2h\,\frac{(x^2-(z+h)^2)}{(x^2+(z+h)^2)^2}\right.$$
$$\left.-4h\,\frac{z(z+h)(3x^2-(z+h)^2)}{(x^2+(z+h)^2)^3}\right) \tag{1}$$

where x is the horizontal distance from the tunnel centreline, z is the depth below ground surface, h is the depth of the tunnel centreline, r_0 is the tunnel radius, ε and δ are the tunnel

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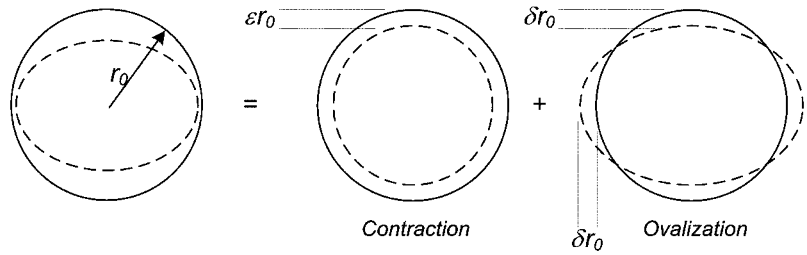

Figure 1. Physical meaning of ε and δ.

contraction and ovalization parameters as shown in Figure 1. When $\delta = 0$ the solution degenerates to that of Sagaseta [6].

The process of determining an upper bound value is based on minimization among different mechanisms of failure, each of which considers different proportions of ovalization to contraction, and different outer slip line positions. As stated, the velocity field for the upper bound solution is assumed to be proportional to the displacement field; i.e. $v = \dot{u} \propto u$. Some justification to this assumption may be found in the centrifuge test results of Mair [7], where the displacement field both at working conditions and at failure resembles the same Gaussian curve. Since the proportionality ratio is cancelled in the upper bound arithmetic, we care only for the ratio of δ/ε and not for their values independently. Since the soil is incompressible, only an outer slip-line, along a flow line, is required to obtain an admissible velocity field.

Figure 2 shows the different zones associated with the terms in the upper-bound energy balance equation

$$\int_V \gamma v_v \, dV + \int_{S_1} \sigma_s v_n \, ds - \int_{S_2} \sigma_t v_n \, ds = \int_V 2 s_u |\dot{\varepsilon}|_{max} \, dV + \int_{S_3} s_u v_t \, ds \tag{2}$$

where γ is the unit weight, v_v is the vertical velocity, σ_s and σ_t are the normal stresses acting on the ground surface plan and the tunnel respectively, v_n and v_t are the normal and tangential velocity along surfaces, $|\dot{\varepsilon}|_{max}$ absolutely largest principal component of the plastic strain rate, and s_u undrained shear strength. The strain increments are calculated analytically from the assumed velocity field using $\dot{\varepsilon}_{ij} = \frac{1}{2}(\partial \dot{u}_i/\partial x_j + \partial \dot{u}_j/\partial x_i)$. Since the volume is incompressible and S_3 is a 'flow line', $\int_{S_1} v_n \, ds = \int_{S_2} v_n \, ds = V\dot{L}$, where $V\dot{L}$ is the rate of volume loss. Equation (2) implicitly involves the use of Tresca's yield criterion

$$f = |\sigma_1 - \sigma_2| + |\sigma_2 - \sigma_3| + |\sigma_3 - \sigma_1| - 4 s_u = 0 \tag{3}$$

where σ_i are the principal stresses.

We follow Davis *et al.* [1] in defining the stability number

$$N = [\sigma_s - \sigma_t + \gamma(C + D/2)]/s_u \tag{4}$$

where C is the tunnel cover depth and D is the tunnel diameter. If an upper bound value is evaluated for $\sigma_s - \sigma_t$ then

$$N = \frac{\int_V 2|\dot{\varepsilon}|_{max} \, dV + \int_{S_3} v_t \, dS - (\beta/D)\int_V v_v \, dV}{V\dot{L}} + \beta \frac{C + D/2}{D} \tag{5}$$

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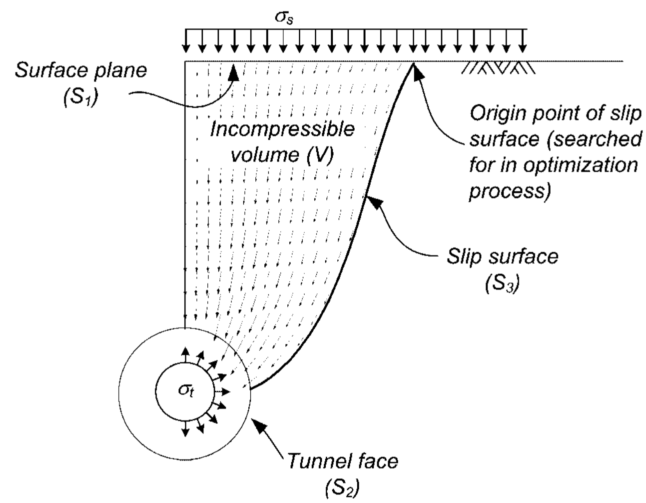

Figure 2. Illustration of mechanism of failure.

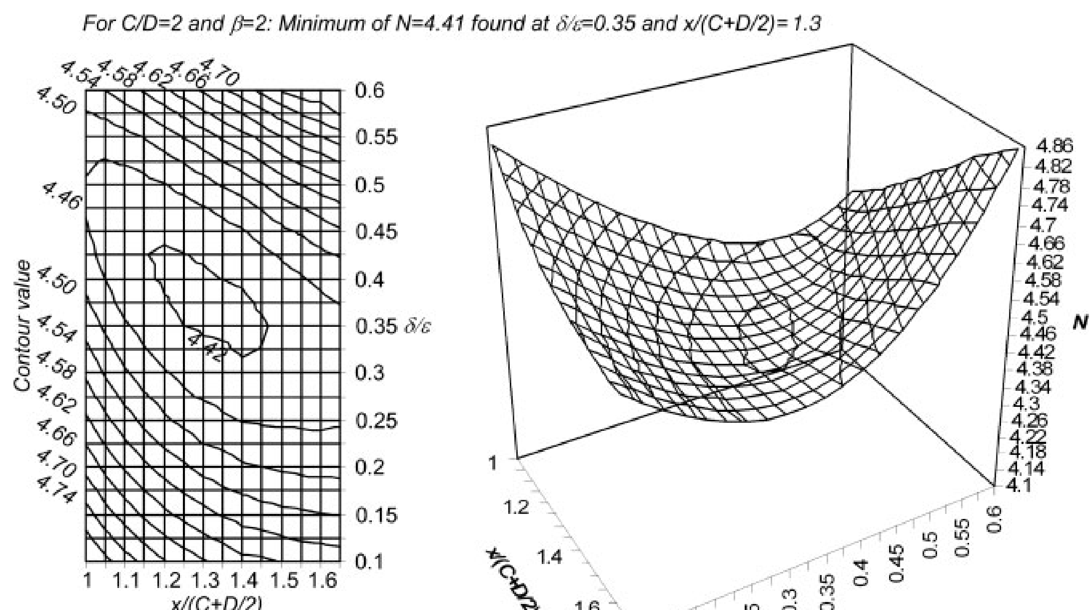

Figure 3. Search field for minimum value of N.

where $\beta = \gamma D/s_u$. The collapse value from the above procedure may be considered a strict upper bound, as the mechanism does not generate any gaps or overlaps anywhere in the soil, and all terms of internal and external work are evaluated and equated. The search for minimum values did not involve any sophisticated minimization algorithm, but a simple response surface by mapping values into a grid. The calculation time for a single point on the

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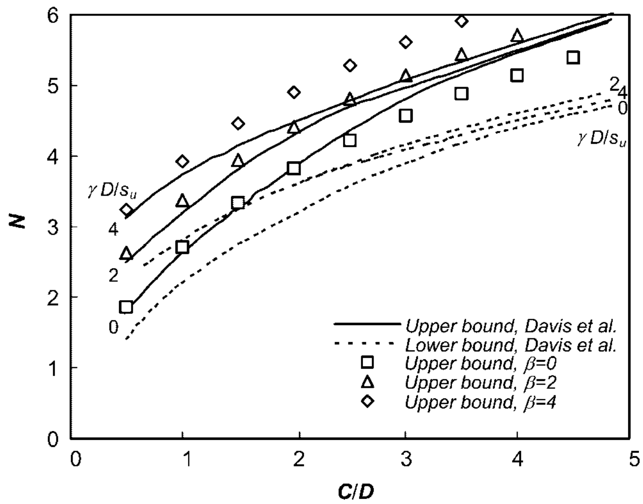

Figure 4. Stability numbers for 2D case.

grid was about 3 s, using Mathematica on a 1.8 GHz Centrino PC. The same search procedure, only on different parameters, was utilized in the solution of the 3D problem presented in the next section.

Figure 3 shows values of N as a function of δ/ε, and the ratio $x/(C+D/2)$ representing the outer slip surface, for the case of $C/D=2$ and $\beta=\gamma D/s_u=2$. Figure 4 shows a comparison between the minimum values obtained from the above procedure and those of Davis *et al.* [1].

Extending the comparison to other collapse mechanisms resulted in: (i) better (lower) upper bound values than that of Osman *et al.* [4] who used a Gaussian settlement trough as an indicator for the velocity field; (ii) slightly better (lower, but by less than 1%) than the seven-variable mechanism of Sloan and Assadi [8] for $\beta=0$ and $C/D=4$, and worse (higher) for $C/D<4$.

3D ANALYSIS

The current style of solution can be extended to the 3D case of face stability associated with tunnel construction. The method of Sagaseta [6] is used to generate the deformation field, using a superposition of sinks and sources. A closed-form solution for the displacement field is derived by integrating the point solution for a sink in an infinite elastic space (and a virtual source placed in mirror symmetry with respect to the required ground surface) over the volume of ground loss into the tunnel. The resulting expression is

$$u_x = -\frac{\xi}{\pi}\frac{hxz}{h^4+2h^2(x^2-z^2)+(x^2-z^2)^2} + \frac{\xi}{4\pi}xy\left(\frac{1}{(x^2+(h-z)^2)\sqrt{x^2+y^2+(h-z)^2}} - \frac{1}{(x^2+(h+z)^2)\sqrt{x^2+y^2+(h+z)^2}}\right)$$

$$u_y = \frac{\xi}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+(h+z)^2}} - \frac{1}{\sqrt{x^2+y^2+(h-z)^2}}\right)$$

$$u_z = \frac{\xi}{2\pi}\frac{h(h^2+x^2-z^2)}{h^4+2h^2(x^2-z^2)+(x^2+z^2)^2} + \frac{\xi}{4\pi}y\left(\frac{z-h}{(x^2+(h-z)^2)\sqrt{x^2+y^2+(h-z)^2}} - \frac{z+h}{(x^2+(h+z)^2)\sqrt{x^2+y^2+(h+z)^2}}\right) \quad (6)$$

This corresponds to cavity collapse within a paved half space (following Sagaseta's terminology). That is, the ground surface is considered to be constrained in the horizontal direction. When $y \to -\infty$ Equation (6) shows that the longitudinal ground movement $u_y = 0$ so

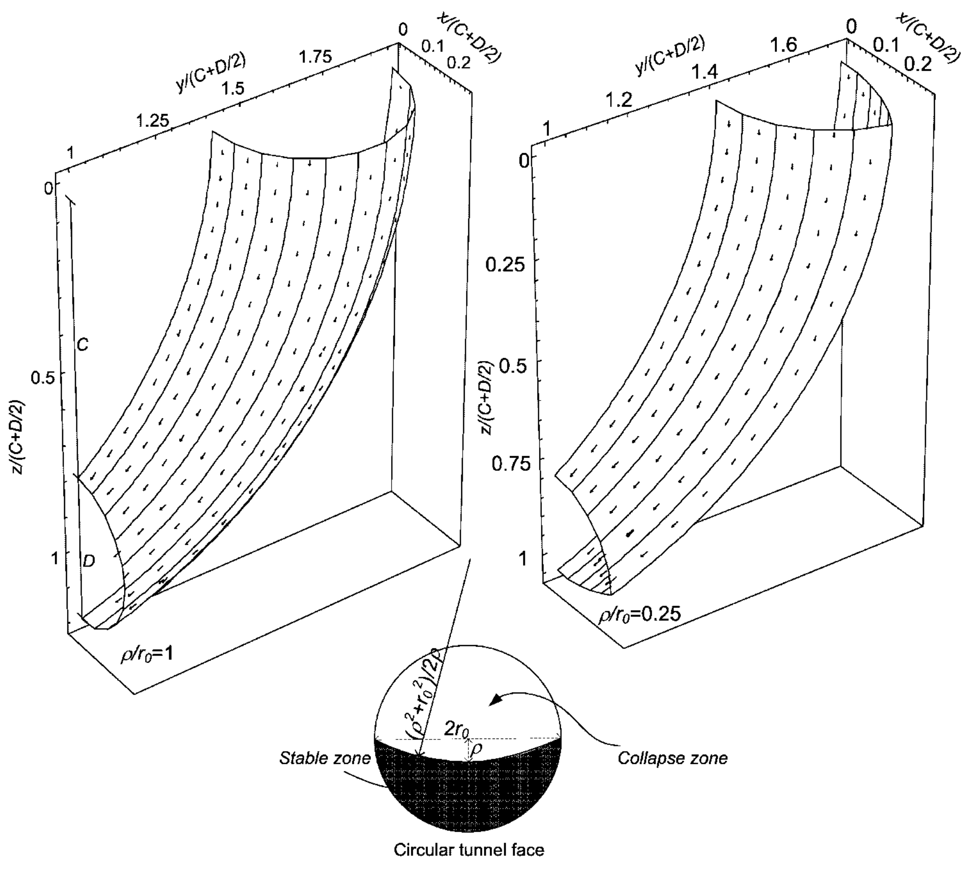

Figure 5. Illustration of isolated collapse body (half a problem), $C/D = 2$.

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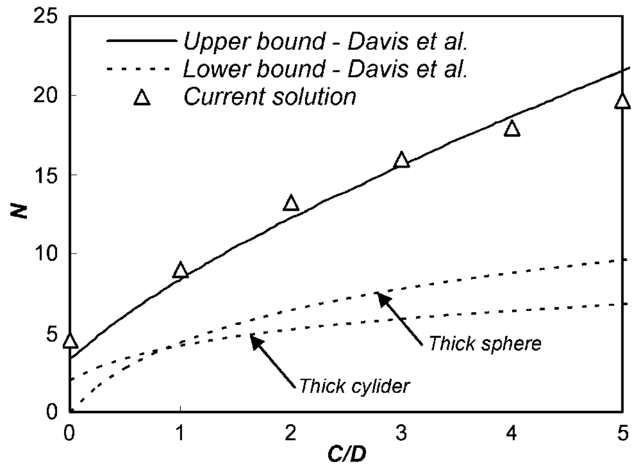

Figure 6. Stability numbers for 3D case.

that a plane strain condition prevails. As with the earlier 2D solution, the value of ξ, or actually $\dot{\xi}$, is not of importance as it is cancelled in the upper bound arithmetic.

Since the solution is singular at the line of sinks, the tunnel face is assumed to be positioned at some distance in front of the line. The search for the minimum upper bound value included a variation of this arbitrary distance. Using the deformation field, a failure body was isolated by a similar procedure as for the 2D case, by choosing flow lines which separate the collapsing body from the rigid soil. The failure mechanisms considered are only for face stability assuming the lining is positioned up to the tunnel face. Partial collapse of the tunnel facing was considered by referring not only to circular face collapse, but to a more general shape (Figure 5). The external surface failure originates from the border of the collapsing face of the tunnel. Figure 5 illustrates the outer boundary of such a collapse body. The strain field inside the collapsing body was derived analytically from differentiating Equation (6).

As before, Tresca yield criterion was used to calculate plastic work. In the minimization procedure it was found that full-face tunnel collapse resulted in the smallest upper bound value. The distance of the tunnel face from the line of sinks varied in the search from 0.1 to 3 diameters. The effect of face location increased with cover depth. The deeper the tunnel, the closer is the optimal face to the line of sinks.

Figure 6 shows a comparison of upper bound values with those previously published for 3D collapse by Davis *et al.* [1]. As in the 2D case, the values for shallow tunnels are higher (worse) than previously published while for deeper tunnels they are lower (better). β had a negligible effect on the resulting values. Consequently, the results may be associated with any value of $\gamma D/s_u$.

CONCLUSIONS

Upper bound calculations for the problem of tunnel excavation were conducted using admissible strain fields derived from the theory of elasticity. The calculation approach entails

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three main stages: (1) Obtaining a closed-form expression for an admissible velocity field by assuming proportionality to an incompressible deformation field derived from elasticity. (2) Isolating a failure body for analysis by identifying a flow line as a slip line. (3) Minimization of energy terms through a search for the optimal collapse body. The upper bound value is a function of the identified flow line which defines the outer boundary of the collapsing body. Since the method involves closed-form expressions of the strain rate field, accurate numerical, or even analytical, integrations of dissipation are possible and quick. Variation of soil properties (density and strength) with depth or location can easily be incorporated. The solution for both the 2D and 3D problems resulted for some cases in lower (better) upper bound values than values previously published by Davis *et al*. [1].

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